

Symplectic structure of electromagnetic duality

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Abstract

We develop an electromagnetic symplectic structure on the space-time manifold by defining a Poisson bracket in terms of an invertible electromagnetic tensor $F_{\mu\nu}$. Moreover, we define electromagnetic symplectic diffeomorphisms by imposing a special condition on the electromagnetic tensor. This condition implies a special constraint on the Levi-Civita tensor. Then, we express geometrically the electromagnetic duality by defining a dual local coordinates system.

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I. INTRODUCTION

In the last twenty years, a field-theoretical Poisson geometry has been developed and was proved to be useful for the study of integrable systems described in terms of ordinary differential equations, partial differential equations and quantum field theory [1].

Symplectic groups have been shown to be powerful material to handle symmetries of quantum field theories: the group $SDiff(S^2)$ of the symplectic transformations on the sphere S^2 is a residual symmetry in relativistic membrane when quantized in the light cone gauge [2]. The reformulation of $D = 4$ $SU(\infty)$ Yang-Mills theory as a Poisson gauge theory on the space $M_6 = M_4 \otimes S^2$, where M_4 is the Minkowski space-time, was based on the equivalence $SDiff(S^2) \sim SU(N \rightarrow \infty)$ [3]. Moreover, the developement of the symplectic structure of harmonic superspace (which is the framework of an off-shell formulation of $D = 4$ $N = 2$ supersymmetric theories) has been proved useful to derive a general form of the symplectic invariant coupling of the Maxwell gauge prepotential [4].

An other interesting application of symplectic geometry in physics was given by Guillemin and Sternberg [5], where they have proved that the electromagnetic field (and charges) determines the symplectic structure of the eight-dimensional phase space of a relativistic particle. this relationship between electromagnetism and space-time is, on one hand a pure relativistic phenomenon and on the other hand a certain kind of duality. Indeed, duality has appeared in different contexts in theoretical physics and can be used as an organizing principle [6]. It means that there is two equivalent descriptions of a model using different fields which are related by Legendre transformations [7]. One early example is the $4D$ duality between electricity and magnetism in Maxwell,s theory that exchanges the weak coupling regime with the strong coupling one. A more sophisticated kind of duality was conjectured by Montonen and Olive [8] to hold for non-abelian gauge theories which interchanges electric charges (related via Noether,s theorem to the existence of symmetries) and magnetic charges which are topological in nature. Seiberg and Witten have produced a supersymmetric version of these ideas [9]. Very recently, Sen has conjectured that such duality implies results about

L^2 -cohomology of monopole moduli spaces.

In this paper we establish a symplectic structure on the space-time manifold via the electromagnetic tensor $F_{\mu\nu}$ of the Maxwell theory. Indeed, an "electromagnetic Poisson structure" on the symplectic electromagnetic manifold (F, M_4, Γ) , where F is the electromagnetic 2-form, M_4 is the space-time manifold and Γ is the stationary surface: the space of all gauge fields that satisfy equations of motion and for which the matrix $(F_{\mu\nu})$ is invertible. Moreover, we derive symplectic diffeomorphisms via a specific constraint on the electromagnetic tensor. Then, we express geometrically the electromagnetic duality by giving a "dual local coordinates system" of the first one where the symplectic electromagnetic structure is defined.

In section 2 we recall the symplectic manifold properties that is endowed with a Poisson bracket structure. In section 3 we construct a symplectic structure where the symplectic form is the electromagnetic tensor. Then, we define electromagnetic symplectic diffeomorphisms by imposing a specific constraint on the electromagnetic tensor which induces a new relation for the Levi-Civita tensor. In section 4 we give a symplectic form to the electromagnetic field equations of motion.. In section 5 we establish a geometrical structure of the electromagnetic duality by expressing the constraint duality in terms of the diffeomorphisms operators. Section 6 is devoted to our conclusion and open problems.

II. SYMPLECTIC STRUCTURE OF A SYMPLECTIC MANIFOLD

An even dimensional manifold M^{2n} endowed with a closed 2-form ω , i.e. $d\omega = 0$, is called a symplectic manifold and the structure given by ω is the symplectic structure. The antisymmetric and non-degenerate bilinear 2-form ω is locally written as [1] :

$$\omega = \frac{1}{2}\omega_{ij}dx^i \wedge dx^j, \quad i, j = 1, \dots, 2n$$

$$\det(\omega_{ij}) \neq 0, \tag{2.1}$$

where $(x^i), i = 1, \dots, 2n$ is a local coordinates system of M^{2n} . This symplectic form defines a skew-symmetric scalar product on the tangent space: for any two vectors $V = (V^i)$ and

$W = (W^i)$ this scalar product is given by

$$\begin{aligned}(V, W) &= \omega_{ij} V^i W^j \\ &= -(W, V).\end{aligned}\tag{2.2}$$

The inverse matrix (ω^{ij}) of (ω_{ij}) is defined by

$$\omega^{ij} \omega_{ij} = \delta_j^i.\tag{2.3}$$

If the symplectic form ω is exact, i.e. $\omega = d\theta$, where $\theta = \theta_k(x) dx^k$ is a 1-form we get

$$\omega_{kl} = \partial_k \theta_l - \partial_l \theta_k.\tag{2.4}$$

The symplectic manifold (M^{2n}, ω) can be interpreted as the phase space of a classical system, where the observables have a natural Poisson-Lie structure [12]. The Hamiltonian vector field $V_f = (V_f^i)$ associated to an observable $f(x)$ is defined by the equation

$$\begin{aligned}i_{v_f} \omega &= -df \\ &\equiv V_f^i \omega_{ij} dx^j\end{aligned}\tag{2.5}$$

from which we get the vector field components:

$$V_f^i = \omega^{ij} \partial_j f.\tag{2.6}$$

This Hamiltonian vector field generates a one-parameter family of local canonical transformations. The Poisson bracket of two observables f and g is given in terms of their gradients

$$\{f, g\} = \omega^{ij} \partial_i f \partial_j g\tag{2.7}$$

and satisfies the following requirements

$$\begin{aligned}\{f, g\} &= -\{g, f\} \\ \{f + g, h\} &= \{f, h\} + \{g, h\} \\ \{fg, h\} &= f\{g, h\} + \{f, h\}g.\end{aligned}\tag{2.8}$$

By using the coordinates (x^i) , eq.(2.7) reduces to

$$\{x^i, x^j\} = \omega^{ij}. \quad (2.9)$$

Furthermore, the Jacobi identity is equivalent to [1]:

$$\partial_k \omega_{ij} + \partial_j \omega_{ki} + \partial_i \omega_{jk} = 0, \forall i, j, k = 1, \dots, 2n \quad (2.10)$$

which is deduced from the equation

$$d\left(\frac{1}{2}\omega_{ij}dx^i \wedge dx^j\right) = 0. \quad (2.11)$$

By definition, a function $f \in C^\infty(M^{2n})$ is a Casimir for the Poisson bracket (2.7) if it belongs to its kernel, i.e. if for any function $g \in C^\infty(M^{2n})$ we have [12]:

$$\{f, g\} = 0. \quad (2.12)$$

The symplectic diffeomorphisms on the manifold M^{2n} which are connected to the identity and denoted by $Diff_0(M^{2n})$ are generated by the operators [13]

$$L_f = \omega^{ij} \partial_j f \partial_i, \quad (2.13)$$

where f is any arbitrary function on M^{2n} . They obey the following algebra

$$[L_f, L_g] = L_{\{f, g\}}. \quad (2.14)$$

III. ELECTROMAGNETIC SYMPLECTIC MANIFOLD

Here, we construct a symplectic structure on the space-time manifold whose symplectic form is the electromagnetic tensor $F_{\mu\nu}$ which verifies all the symplectic properties of section 2. In this framework we give a symplectic expression for the Maxwell's equations and for the electromagnetic Lagrangian [14].

A. Electromagnetic Poisson structure

Let us consider the electromagnetic field strength in the four Minkowski space-time:

$$F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu, \quad (3.1)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is the antisymmetric tensor and A_μ is the electromagnetic field. Furthermore, it is well known [15,16] that the 2-form F is closed and exact:

$$\begin{aligned} dF &= 0, \\ F &= dA, \end{aligned} \quad (3.2)$$

where A is the 1-form gauge field; $A(x) = A_\mu(x)dx^\mu$.

Now, we define the electromagnetic symplectic structure as follows: For any functions $f, g \in C^\infty(M_4)$ the Poisson bracket is given by

$$\{f, g\} = F^{\mu\nu} \partial_\mu f \partial_\nu g. \quad (3.3)$$

In particular, in the local coordinates system (x^μ) , $\mu = 0, \dots, 3$, we have

$$\{x^\mu, x^\nu\} = F^{\mu\nu}. \quad (3.4)$$

This equation (3.4) can be seen as a relationship between space-time and electromagnetism in order to be endowed with a symplectic structure. The corresponding properties are discussed in the next subsection.

The gauge fields considered here are elements of gauge orbits denoted by Γ [11], where $\det F_{\mu\nu} \neq 0$ and does not contain pure gauge fields, i.e. $A_\mu = g^{-1}\partial_\mu g \Leftrightarrow F_{\mu\nu} = 0$ [15]. On the other hand, the determinant of the antisymmetric tensor $F_{\mu\nu}$ is given by

$$\begin{aligned} \det(F_{\mu\nu}) &= \left(\frac{1}{8}\varepsilon^{\mu\nu\sigma\lambda}F_{\mu\nu}F_{\sigma\lambda}\right)^2 \\ &= \left(\frac{1}{4}F_{\mu\nu} * F^{\mu\nu}\right)^2, \end{aligned} \quad (3.5)$$

where $*F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\sigma\lambda}F_{\sigma\lambda}$ is the dual tensor of $F^{\mu\nu}$ in four dimensional space-time. So, we consider the electromagnetic theory with a non vanishing topological charge

$$\begin{aligned} Q &= \frac{1}{8\pi^2} \int F \wedge F \\ &= \frac{-1}{4} \int C_2(A), \end{aligned} \quad (3.6)$$

where $C_2(A)$ is the second Chern class [15]. Furthermore, we define the inverse of the matrix $(F^{\mu\nu})$ by the relation

$$\begin{aligned} F_{\mu\nu} *F^{\nu\sigma} &= \frac{1}{4}\delta_\mu^\sigma (F_{\mu\nu} *F^{\nu\mu}) \\ &= \delta_\mu^\sigma \sqrt{\det(F_{\mu\nu})} \end{aligned} \quad (3.7)$$

which can be rewritten as

$$F_{\mu\nu} \frac{*F^{\nu\sigma}}{\sqrt{\det(F_{\mu\nu})}} = \delta_\mu^\sigma. \quad (3.8)$$

The Jacobi identity of this symplectic structure is the Bianchi identity verified by the electromagnetic tensor:

$$\partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma} = 0 \quad (3.9)$$

which is induced from the closure of the electromagnetic 2-form, i.e. $dF = 0$.

B. Properties of the electromagnetic symplectic structure

The electromagnetic field is a function of the space-time coordinates: $A_\mu = A_\mu(x)$. Then, Let us consider the particular solution for A_μ that is,

$$A^\mu(x) = x^\mu. \quad (3.10)$$

In this case the electromagnetic tensor $F_{\mu\nu}$ vanishes and the eq.(3.4) leads to the constraint

$$\{x^\mu, x^\nu\} = 0 \quad (3.11)$$

which means that these particular solutions (3.10) are observable Casimirs of the electromagnetic symplectic structure. It means also that the absence of electromagnetism is expressed by the triviality of the Poisson bracket.

Now, using eq.(3.3) for the electromagnetic field A^μ , i.e.

$$\{A^\mu, A^\nu\} = F^{\sigma\lambda} \partial_\sigma A^\mu \partial_\lambda A^\nu \quad (3.12)$$

we get

$$\{A^\mu, A^\nu\} = L^\mu(A^\nu) - L^\nu(A^\mu) + A(A^\mu) \partial_\lambda \partial^\lambda A^\nu - A(A^\nu) \partial_\lambda \partial^\lambda A^\mu, \quad (3.13)$$

where

$$L^\mu \equiv \partial^\lambda (A^\sigma \partial_\lambda A^\mu) \partial_\sigma,$$

$$A(A^\mu) \equiv A^\sigma \partial_\sigma A^\mu.$$

Then, by considering equations of motion of the gauge field in the Lorentz gauge ($\partial_\mu A^\mu = 0$), the relation (3.13) reduces to

$$\begin{aligned} \{A^\mu, A^\nu\} &= L^\mu(A^\nu) - L^\nu(A^\mu) \\ &\equiv [L, A]^{\mu\leftrightarrow\nu} \\ &\equiv -L^{\nu\mu} \end{aligned} \quad (3.14)$$

The eq.(3.14) tells us that the electromagnetic symplectic structure is realized on gauge orbits as a commutation relation. On the other hand, if we consider the particular solution for A^μ eq.(3.10), the generators L^μ reduces to

$$L^\mu = \partial^\mu, \quad (3.15)$$

which are tangent vectors on the space-time manifold, and eq.(3.14) coincides with eq.(3.11).

For the general case we get the following commutation relations:

$$[L^\mu, L^\nu] = \{\partial_\sigma (\partial^\lambda A^\sigma \partial^\rho A^m \partial_\rho A^\nu) \partial_\lambda A^\mu - \partial_\sigma (\partial^\lambda A^\sigma \partial^\rho A^m \partial_\rho A^\mu) \partial_\lambda A^\nu\} \partial_m \quad (3.16)$$

C. Electromagnetic symplectic diffeomorphisms

In analogy with the definition of symplectic diffeomorphisms given in section 2 we define the generators of electromagnetic symplectic diffeomorphisms as follows:

$$L_f = F^{\mu\nu} \partial_\nu f \partial_\mu, \quad (3.17)$$

where f is an arbitrary function of x . In particular, the associated diffeomorphism generators to local coordinates (x^λ) are given by

$$L_{x^\lambda} = F^{\mu\lambda} \partial_\mu. \quad (3.18)$$

Furthermore, we recover the relation (3.4) as follows:

$$\begin{aligned} L_{x^\mu}(x^\nu) &= F^{\nu\mu} \\ &= \{x^\nu, x^\mu\} \end{aligned} \quad (3.19)$$

So, we can verify that

$$L_{x^\lambda}(F^{\mu\nu}) = \{F^{\mu\nu}, x^\lambda\}, \quad (3.20)$$

and

$$L_{x^\lambda}(A^\mu) = \{A^\mu, x^\lambda\}. \quad (3.21)$$

So, the generators L_{x^λ} are realized as a Poisson bracket on the observables:

$$\begin{aligned} L_{x^\lambda} &= \{x^\mu, x^\lambda\} \partial_\mu \\ &= \{, x^\lambda\}. \end{aligned} \quad (3.22)$$

In order to have an electromagnetic symplectic diffeomorphism algebra we impose the following commutation relations

$$\begin{aligned} [L_{x^\mu}, L_{x^\nu}] &= L_{\{x^\mu, x^\nu\}} \\ &= L_{F^{\mu\nu}} \end{aligned} \quad (3.23)$$

which can be rewritten, by considering the eq.(3.22), as

$$[\{, x^\mu\}, \{, x^\nu\}] = \{, \{x^\mu, x^\nu\}\}. \quad (3.24)$$

This means that the electromagnetic symplectic diffeomorphisms algebra is realized as the Poisson bracket structure.

After some developments we find

$$L_{F^{\mu\nu}} = \partial_\lambda (F^{\sigma\lambda} F^{\mu\nu}) \partial_\sigma \quad (3.25)$$

$$[L_{x^\mu}, L_{x^\nu}] = \partial_\lambda (F^{\lambda\mu} F^{\sigma\nu} - F^{\lambda\nu} F^{\sigma\mu}) \partial_\sigma. \quad (3.26)$$

Then, from eq.(3.23) we get the following constraint on the electromagnetic tensor up to a divergence term:

$$F^{\lambda\mu} F^{\sigma\nu} - F^{\lambda\nu} F^{\sigma\mu} = F^{\sigma\lambda} F^{\mu\nu}. \quad (3.27)$$

It is easy to verify that the eq.(3.27) is invariant under a circular permutation of indices $(\lambda, \mu, \sigma, \nu)$. On the other hand, by expressing eq.(3.27) in terms of the dual tensor $*F^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\sigma\lambda} F_{\sigma\lambda}$ we get a special constraint on the Levi-Civita tensor:

$$\varepsilon^{\lambda\mu mn} \varepsilon^{\sigma\nu pq} - \varepsilon^{\lambda\nu mn} \varepsilon^{\sigma\mu pq} = \varepsilon^{\sigma\lambda mn} \varepsilon^{\mu\nu pq}. \quad (3.28)$$

Furthermore, by inserting Maxwell's field equations of motion $(\partial_\mu F^{\mu\lambda} = 0)$ in eq.(3.27) we find the following equation

$$L_{x^\mu} (F^{\sigma\nu}) - L_{x^\nu} (F^{\sigma\mu}) = L_{x^\sigma} (F^{\mu\nu}) \quad (3.29)$$

which can be expressed, by using eq.(3.20), as

$$\{F^{\sigma\nu}, x^\mu\} - \{F^{\sigma\mu}, x^\nu\} = \{F^{\mu\nu}, x^\sigma\}. \quad (3.30)$$

Also, one check that eq.(3.30) is invariant under any circular permutation of the indices (σ, ν, μ) and gives the same equation (3.28) by replacing $F^{\mu\nu}$ by its expression in terms of $*F^{\mu\nu}$. On the other hand, eqs.(3.27,29) give the following relation

$$\varepsilon^{\sigma\nu mn}\eta^{\mu\rho} - \varepsilon^{\sigma\mu mn}\eta^{\nu\rho} = \varepsilon^{\mu\nu mn}\eta^{\sigma\rho}, \quad (3.31)$$

where $\eta^{\mu\nu}$ is the Minkowski space-time metric. This means that eqs.(3.27,29) are equivalent.

Indeed, by inserting again the equations of motion in eq.(3.29) we get

$$F^{\lambda\mu}\partial_\mu\partial_\lambda F^{\sigma\nu} - \partial_\mu F^{\lambda\nu}\partial_\lambda F^{\sigma\mu} = \partial_\mu F^{\sigma\lambda}\partial_\lambda F^{\mu\nu} \quad (3.32)$$

which reduces to

$$\partial_\mu\partial_\lambda(F^{\lambda\mu}F^{\sigma\nu} - F^{\lambda\nu}F^{\sigma\mu} - F^{\sigma\lambda}F^{\mu\nu}) = 0 \quad (3.33)$$

and gives again eq.(3.27) up to the term $\Lambda^{\mu\lambda} = x^\lambda C^\mu + x^\mu C^\lambda + D^{\mu\lambda}$ such that, $\partial_\lambda D^{\mu\lambda} = 0 = \partial_\lambda D^{\mu\lambda}$ and C^μ is a constant vector. Hence, the eq.(3.27) can be generated, up to a quadratic term in x^λ , by inserting the equations of motion in eq.(3.33) and so on.

IV. SYMPLECTIC FORM OF THE ELECTROMAGNETIC FIELD EQUATIONS OF MOTION

The equations of motion of the electromagnetic field are given by

$$\partial_\mu F^{\mu\nu} = 0. \quad (4.1)$$

Their dual form is the Bianchi identity

$$\partial_\mu {}^*F^{\mu\nu} = 0 \quad (4.2)$$

with ${}^*F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\sigma\lambda}F_{\sigma\lambda}$ is the dual of the electromagnetic tensor. Furthermore, by using the Poisson bracket expression (eq.(3.4)) in eq.(4.1) we get the geometrical form of the equations of motion:

$$\partial_\mu \{x^\mu, x^\nu\} = 0 \quad (4.3)$$

which can be deduced from the general formula

$$(2 + x^\mu\partial_\mu)\{f, x^\lambda\} = \partial_\sigma\{x^\sigma f, x^\lambda\}, \quad (4.4)$$

where $f \in C^\infty(M_4)$, by setting $f = 1$. The latter equation is established by using eqs.(3.21) and (2.8). Equivalently, eq.(4.3) can be expressed as follows

$$[\partial_\lambda, L_{x^\lambda}] = 0 \leftrightarrow \partial_\lambda L_{x^\lambda} = 0, \quad (4.5)$$

where L_{x^λ} are the symplectic diffeomorphism generators given in eq.(3.22), because L_{x^λ} is proportional to $\partial_\lambda : L_{x^\lambda} = F^{\sigma\lambda} \partial_\sigma$. However, the Bianchi identity can be written as

$$[\partial_\lambda, {}^* L_{x^\lambda}] = 0 \leftrightarrow \partial_\lambda {}^* L_{x^\lambda} = 0, \quad (4.6)$$

where ${}^* L_{x^\lambda} \equiv {}^* F^{\sigma\lambda} \partial_\sigma$. Furthermore, one can verify that

$$\begin{aligned} L_{x^\mu}({}^* F_{\mu\nu}) &= {}^* L_{x^\mu}(F_{\mu\nu}), \\ &= \partial_\nu(\sqrt{\det(F_{\mu\nu})}) \end{aligned} \quad (4.7)$$

(by using eq.(3.8)) and

$$[L_{x^\lambda}, {}^* L_{x^\sigma}] = \partial_m(F^{m\lambda} {}^* F^{\mu\sigma} - F^{\mu\lambda} {}^* F^{m\sigma}) \partial_\mu \quad (4.8)$$

which gives

$$[L_{x^\lambda}, {}^* L_{x^\lambda}] = 0. \quad (4.9)$$

The last equation can be understood from eqs.(4.5,6). Then, the equations of motion and the Bianchi identity are realized as the electromagnetic symplectic diffeomorphisms that commute with the partial derivative.

Now, let us consider the Maxwell's action

$$S_A = \frac{-1}{4} \int_{M^4} d^4x F_{\mu\nu} F^{\mu\nu}. \quad (4.10)$$

By replacing $F_{\mu\nu}$ by its expression (eq.(3.4)), the action (4.10) becomes

$$S_A = \frac{-1}{4} \int_{M^4} \{x_\mu, x_\nu\} \{x^\mu, x^\nu\}. \quad (4.11)$$

It is analogous to the string action given by Schild [14] which is the square of the usual Lagrangian representing the surface area element, and hence does not have a purely geometrical meaning. Furthermore, by using the following expression for the Lagrangian

$$\begin{aligned} -2\mathcal{L}_A &= -A_\lambda \partial_\mu F^{\mu\lambda} + \partial_\mu (A_\lambda F^{\mu\lambda}) \\ &= \{A_\lambda, x^\lambda\}, \end{aligned} \quad (4.12)$$

the action (4.11) takes the simple form

$$S_A = \frac{-1}{2} \int_{M^4} \{A_\lambda, x^\lambda\} \quad (4.13)$$

which can be rewritten (when using eq.(3.21)) as

$$S_A = \frac{-1}{2} \int_{M^4} L_{x^\lambda}(A_\lambda). \quad (4.14)$$

It is easy to get from eqs.(4.11,14) the following identity for the symplectic diffeomorphism generators

$$L_{x^\nu}(x^\mu) L_{x_\nu}(x_\mu) = 2L_{x^\lambda}(A_\lambda). \quad (4.15)$$

In the expression (4.13) of the action gauge invariance is manifest. Indeed, let us consider a $U(1)$ gauge transformation for the gauge field,

$$A_\lambda \rightarrow A'_\lambda = A_\lambda + \partial_\lambda \quad (4.16)$$

then,

$$\{A'_\lambda, x^\lambda\} = \{A_\lambda, x^\lambda\} \quad (4.17)$$

because we have

$$\begin{aligned} \{\partial_\lambda \Lambda, x^\lambda\} &= F^{\sigma\lambda} \partial_\sigma \partial_\lambda \Lambda \\ &= 0 \end{aligned} \quad (4.18)$$

V. SYMPLECTIC STRUCTURE OF THE ELECTROMAGNETIC DUALITY

In the ref.[7] the authors have shown that the electromagnetic duality in four dimensional space-time is a vector-vector duality: the gauge field A_μ is interchanged with the vector field Λ_μ such that the variation of the parent action

$$\begin{aligned} S_{F,\Lambda} &= \int_{M^4} d^4x \left(\frac{-1}{4} F_{\mu\nu} F^{\mu\nu} + \Lambda_\mu \partial_\nu {}^* F^{\nu\mu} \right) \\ &= \int_{M^4} d^4x \left(\frac{-1}{2} L_{x^\lambda}(A_\lambda) + \Lambda_\mu \partial_\nu {}^* L_{x^\mu}(x^\nu) \right) \end{aligned} \quad (5.1)$$

with respect to Λ_μ gives the Bianchi identity $(\partial_\nu {}^* L_{x^\mu}(x^\nu))$ and the action for the electromagnetic field. However, the variation of the same action with respect to $F_{\mu\nu}$ gives the equation

$$F^{\mu\nu} = -\partial_\rho \Lambda_\sigma \varepsilon^{\rho\sigma\mu\nu} \equiv -{}^* G^{\mu\nu} \quad (5.2)$$

which can be rewritten in terms of the operators L_{x^μ} and ${}^* L_{x^\mu}$ as follows:

$$L_{x^\lambda}(A_\lambda) = -{}^* L_{x^\lambda}(\Lambda_\lambda) \quad (5.3)$$

and then leads to the action of the field Λ_μ

$$S_\Lambda = -\frac{1}{2} \int_{M^4} d^4x {}^* L_{x^\lambda}(\Lambda). \quad (5.4)$$

It is easy to verify that from eqs.((4.14),(5.3,4)) we get

$$\mathcal{L}_A = -\mathcal{L}_\Lambda \quad (5.5)$$

which means that the two actions describe the same physics but with different descriptions. This is the electromagnetic duality. Furthermore, if we introduce the constant couplings g^2 for A_μ and g'^2 for Λ_μ we obtain

$$g'^2 = \frac{1}{g^2}. \quad (5.6)$$

In fact, the eq.(5.3) reflects the duality invariance between A_λ and Λ_λ as follows: We define

$$\begin{aligned}
*(A_\lambda) &\equiv \Lambda_\lambda(\text{dual of } A_\lambda) \\
*(L_{x^\lambda}) &\equiv *L_{x^\lambda}(\text{dual of } L_{x^\lambda}),
\end{aligned} \tag{5.7}$$

and then we have

$$\begin{aligned}
*(L_{x^\lambda}(A_\lambda)) &= *L_{x^\lambda}(\Lambda_\lambda) \\
&= -L_{x^\lambda}(A_\lambda)
\end{aligned} \tag{5.8}$$

which means that the eq.(5.3) is invariant with respect to duality transformations. On the other hand, let us consider an other local coordinates system (σ^μ) of the space-time manifold defined by

$$\{\sigma^\mu, \sigma^\nu\} = G^{\mu\nu} \equiv \partial^{[\mu} \Lambda^{\nu]} \tag{5.9}$$

to which we can associate the new symplectic diffeomorphism operators

$$\begin{aligned}
d_{\sigma^\mu}(\sigma^\nu) &\equiv -G^{\mu\nu} \\
*d_{\sigma^\mu}(\sigma^\nu) &\equiv -*G^{\mu\nu}.
\end{aligned} \tag{5.10}$$

Then, the duality condition (5.2) (or (5.3)) is rewritten as

$$\begin{aligned}
*L_{x^\mu}(x^\nu) &= d_{\sigma^\mu}(\sigma^\nu) \\
L_{x^\mu}(x^\nu) &= -*d_{\sigma^\mu}(\sigma^\nu)
\end{aligned} \tag{5.11}$$

where,

$$\begin{aligned}
*L_{x^\mu}(x^\nu) &= \frac{1}{2} \varepsilon^{\mu\nu mn} L_{x_m}(x_n) \\
*d_{\sigma^\mu}(\sigma^\nu) &= \frac{1}{2} \varepsilon^{\mu\nu mn} d_{\sigma_m}(\sigma_n).
\end{aligned} \tag{5.12}$$

the constraint (5.11) (and its dual) expresses the fact that the local system (x^μ) , where the electromagnetic tensor is the electromagnetic symplectic 2-form, is dual to the local one (σ^μ) , where the tensor $G^{\mu\nu}$ defines a symplectic structure dual to the electromagnetic one. Furthermore, the constraint (5.11) can be expressed in terms of local coordinates by the relation

$$\varepsilon_{\mu\nu\lambda\rho} = \partial_\mu \sigma_\lambda \partial_\nu \sigma_\rho, \quad (5.13)$$

where $\sigma^\mu(x)$ are functions of the coordinates (x^μ) and $\partial_\mu = \frac{\partial}{\partial x^\mu}$. This is the geometrical expression of the electromagnetic duality. It can be expressed as a relationship between the operators L_{x^λ} and d_{σ^λ} as follows:

$$\frac{2}{3} \varepsilon^{\mu\nu mn} = \frac{\varepsilon^{\mu\nu\lambda\rho}}{\sqrt{\det(F_{\mu\nu})}} d_{\sigma_\rho}(\sigma_\lambda) L_{x^n}(x^m), \quad (5.14)$$

where

$$\det(F_{\mu\nu}) \neq 0 \quad (5.15)$$

or

$$\begin{aligned} \frac{2}{3} \varepsilon^{\mu\nu mn} &= \frac{\varepsilon^{\mu\nu\lambda\rho}}{\sqrt{\det(F_{\mu\nu})}} d_{\sigma_\rho}(\sigma_\lambda) {}^* d_{\sigma^m}(\sigma^n), \\ \frac{2}{3} \varepsilon^{\mu\nu mn} &= \frac{\varepsilon^{\mu\nu\lambda\rho}}{\sqrt{\det(F_{\mu\nu})}} {}^* L_{x_\rho}(x_\lambda) L_{x^n}(x^m). \end{aligned} \quad (5.16)$$

To derive the equations of motion and the Bianchi identity of the two sectors (A_μ, Λ_μ) we consider the constraint (5.10). Indeed, the equations of motion for A_μ are given by

$$\partial_\mu L_{x^\mu}(x^\nu) = 0, \quad (5.17)$$

and are equivalent to the equations

$$- \partial_\mu {}^* d_{\sigma^\mu}(\sigma^\nu) = 0 \quad (5.18)$$

that express the Bianchi identity of the field Λ_μ . Inversely the equations

$$\partial_\mu {}^* L_{x^\mu}(x^\nu) = 0 \quad (5.19)$$

express the Bianchi identity of A_μ and are equivalent to the equations of motion of Λ_μ :

$$\partial_\mu d_{\sigma^\mu}(\sigma^\nu). \quad (5.20)$$

Here, by considering the duality transformations (eq.(5.3)) between two local systems of the space-time manifold, the electromagnetic symplectic structure transforms into the vectorial

symplectic structure and vice-versa: the gauge field in one local system transforms into a vector field (dual to the gauge field) defined in another local system dual to the first (with respect to eq.(5.3)). Inversly, considering any local system, where the vectorial symplectic structure is defined, we can determine the local system, where the electromagnetic symplectic structure is setting. In other words, duality transformations correspond to a change of coordinates associated to a special diffeomorphisms in the sens of eq.(5.3).

VI. CONCLUSION

An electromagnetic symplectic structure is established on the space-time via the definition of a Poisson bracket in terms of the invertible electromagnetic tensor. furthermore, the definition of the electromagnetic symplectic diffeomorphisms enables us to interpret geometrically the electromagnetic duality, via the duality constraint, by considering a dual local system of coordinates. Then, duality transformations are understood as special diffeomorphisms on the space-time.

It is interesting to generalize our formalism to the Yang-Mills thoery and to its $N = 2$ supersymmetric version where duality has been proved recently [9].

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